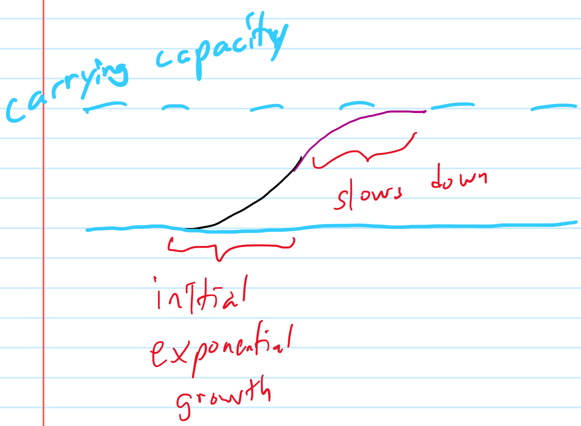
2.6a approximate logistic equation

Monday, February 1, 2021 6:51 AM

Suppose we introduce an organism into an environment.

Initially, might reproduce exponentially, but eventually uses up available resources, at which point the growth slows down.



"S-shaped" sigmoid curve.

$$\frac{dy}{dt} = ay\left(1 - \frac{y}{K}\right)$$ } logistic equation

a: intrinsic growth rate

↳ $K > 0$ is carrying capacity

For now, let's approximate the logistic differential equation via a difference eq.

Note $$\frac{dy}{dt} \approx \frac{y(t+\Delta t) - y(t)}{\Delta t} = ay(t)\left(1 - \frac{y(t)}{K}\right)$$

$$\Rightarrow y(t+\Delta t) = y(t) + a\,\Delta t\, y(t)\left(1 - \frac{y(t)}{K}\right)$$

Assume $\Delta t = 1$.

$$y_{t+1} = y_t + a\,y_t\left(1 - \frac{y_t}{K}\right) = (1+a)\,y_t - \frac{a y_t^2}{K}$$

a: exp. growth param.

K: carrying capacity K

$\frac{a y_t^2}{K}$: limit to population growth dependent on density.

Change variables: $x_t = \frac{a y_t}{K(1+a)}$, $r = 1 + a$

$$x_{t+1} = \frac{a}{K(1+a)} y_{t+1} = \frac{a}{K} \cdot y_t - \frac{a^2}{K(1+a)} y_t^2$$

$$x_{t+1} = \frac{a}{k(1+a)} y_{t+1} = \frac{a}{k} \cdot y_t - \frac{a^2}{k^2(1+a)} y_t^2$$

$$= \frac{a y_t}{k} \left(1 - \frac{a y_t}{k(1+a)} \right) = \frac{a y_t}{k} (1 - x_t)$$

$$= (1+a) x_t (1 - x_t)$$

$$\boxed{x_{t+1} = r x_t (1 - x_t)}$$

Only one parameter r.

dimensionless form discrete logistic equation

Let's ensure things are nonnegative.

Assume $0 < r < 4$ and $0 \le x_0 < 1$.

$(f(x) = r x (1-x))$

$$\max_{x \in [0,1]} f(x) = f\left(\tfrac{1}{2}\right) = \frac{r}{4}.$$

We chose $0 < r < 4$ because if $r > 4$, then $f\left(\tfrac{1}{2}\right) > 1$ and $f\left(f\left(\tfrac{1}{2}\right)\right) < 0$.

By choosing $0 < r < 4$ and $0 \le x < 1$, $f : [0,1) \longrightarrow [0,1)$.

Also, note $f(0) = f(1) = 0$.

With a little work, can find equilibrium solutions

$$\bar{x} = 0 \quad \text{and} \quad \bar{x} = \frac{r-1}{r}.$$

Stability analysis

If $0 < r < 1$, then $\bar{x} = \frac{r-1}{r} < 0$, so 0 is only nonnegative fixed pt.

$$f(x) = r x (1-x)$$
$$f'(x) = r - 2 r x$$

$f'(0) = r$, so 0 is a locally asymptotically stable fixed pt.

If $1 < r < 4$, both $\bar{x} = 0$ and $\bar{x} = \frac{r-1}{r}$ are pos. fixed pts.

If $1 < r < 4$, both $\bar{x} = 0$ and $\bar{x} = \frac{r-1}{r}$ are pos. fixed pts.

$f'(0) = r > 1$, so 0 is unstable

$f'\left(\frac{r-1}{r}\right) = r - 2r\left(\frac{r-1}{r}\right) = r - 2(r-1) = -r + 2.$

Thus, $\frac{r-1}{r}$ is stable when $1 < r < 3$

unstable when $r > 3$.

Aside: This change in stability at $r = 1$ and $r = 3$ are bifurcations, which we will discuss and analyze next time.